AIAA JOURNAL
Vol. 46, No. 11, November 2008

Probabilistic Solution of Nonlinear Oscillators Under External and Parametric Poisson Impulses

G. K. Er,* H. T. Zhu,† V. P. Iu,‡ and K. P. Kou§
University of Macau, Macao SAR, People's Republic of China

DOI: 10.2514/1.36556

This paper deals with the stationary probability density function solution of the dynamic response for nonlinear oscillators excited by Poisson impulses. The probability density function solution is governed by the generalized Fokker–Planck–Kolmogorov equation. The exponential-polynomial closure method is used to obtain an approximate solution to the generalized Fokker–Planck–Kolmogorov equation. To evaluate the effectiveness of the exponential-polynomial closure method in the case of Poisson excitations, Duffing oscillators, van der Pol oscillators, and an additional nonlinear oscillator excited by external and parametric Poisson impulses with different levels of nonlinearity in stiffness and damping are studied. The numerical results show good agreement with the probability distribution obtained with Monte Carlo simulations including the tail regions, which is of significance for reliability analyses.

Nomenclature

a	=	coefficient vector of polynomial Q_n
c	=	normalization constant
$E[Y^2]$	=	mean square of impulse amplitude
h_0, h_j	=	deterministic system functions
i, j, k	=	index
N	=	sample size
$N(T)$	=	total number of pulses in the time interval $(-\infty, T]$
N_p	=	number of unknown parameters
n	=	polynomial order
P	=	probability
p	=	exact probability density function
$\tilde{p}$	=	approximate probability density function
p_f	=	probability to be estimated
$\bar{p}_f$	=	estimated value of p_f
Q_n	=	polynomial of x_1, x_2
$W_j(t)$	=	jth Poisson impulse excitation
X, $\dot{X}$	=	random displacement and velocity
x_1, x_2	=	deterministic displacement and velocity
Y_1, Y_2, Y_3	=	impulse amplitude component
Y_{jk}	=	kth impulse amplitude component for jth Poisson white noise
β_1, β_2	=	scale factor
Δ	=	residual function
$\delta(t)$	=	Dirac delta function
ε	=	error tolerance
ε_1, ε_2	=	nonlinearity parameters
ζ	=	damping ratio
λ, λ_j	=	impulse arrival rate
τ_k	=	arrival time
ω_0	=	angular frequency
$1-\delta$	=	confidence level

Received 9 January 2008; revision received 3 July 2008; accepted for publication 8 August 2008.

*Associate Professor, Department of Civil and Environmental Engineering; gker@umac.mo.

†Ph.D. Candidate, Department of Civil and Environmental Engineering; ya57401@umac.mo.

‡Professor, Department of Civil and Environmental Engineering; vaipaniu@umac.mo.

§Associate Professor, Department of Civil and Environmental Engineering; kpkou@umac.mo.

I. Introduction

MANY engineering problems can be described with nonlinear stochastic oscillators subjected to stochastic excitations. The excitation is often modeled as Gaussian white noise, which represents a continuous stochastic process. Under the action of Gaussian white noise, the probability density function (PDF) of the response of these oscillators is governed by the well-known Fokker–Planck–Kolmogorov (FPK) equation. The stochastic responses of a buckled beam [1], fatigue crack propagation [2,3], and evaporating droplets dispersed in isotropic turbulence [4] were studied by using the FPK equation. However, Gaussian white noise is not always appropriate for describing some discrete stochastic processes, and extensive efforts have been devoted to the investigation of the non-Gaussian processes. In particular, the Poisson impulse (or Poisson white noise) attracts much attention in stochastic mechanics as a typical non-Gaussian process [5]. It represents a discrete excitation process as a sequence of independent, identically distributed pulses arriving at random times. Corresponding to this type of excitation, the FPK equation is generalized [6–9]. However, solving the generalized FPK equations has been a challenge in the last few decades. Only a few exact stationary PDF solutions of highly restricted systems were obtained by an inverse solution procedure [10–12]. Therefore, numerical methods have been proposed in the past few decades to obtain approximate PDF solutions. Roberts [6] outlined a perturbation scheme for the PDF solutions and applied the scheme to a linear system under external Poisson impulses. Subsequently, Cai and Lin [7] improved the perturbation technique in which an initial solution was adopted using the exact stationary solutions for the case of Gaussian white noise. Alternatively, Köylüoğlu et al. [13] employed a Petrov–Galerkin technique to solve the generalized FPK equation. Later, Köylüoğlu et al. [14,15] also proposed a cell-to-cell mapping (or path integration) technique for obtaining the PDF solutions for the case of external Poisson impulses. The effectiveness of these two approaches was briefly discussed by Iwankiewicz and Nielsen [16]. The Petrov–Galerkin technique is suitable for the impulse with a high arrival rate, whereas the cell-to-cell mapping technique is effective for the impulse with a low arrival rate. Recently, Wojtkiewicz et al. [17,18] adopted a finite difference method to solve the generalized FPK equation for the case of external Gaussian and Poisson white noises. However, the values of the PDF solution may be negative in the tail regions. Equivalent linearization (EQL) procedures [19–23] and cumulant-neglect closure procedures [24–26] were also investigated for statistical moments for the case of Poisson white noise. The EQL procedures are suitable for the oscillators with slight nonlinearity and the impulse with a high arrival rate. For the nonlinear oscillators excited by parametric Poisson excitation, Proppe [23] applied several EQL techniques originally developed for parametric Gaussian excitation to the Poisson case. The accuracy obtained by different linearization procedures is similar for external excitation but inconsistent for parametric excitation. For the cumulant-neglect closure procedure, either fourth- or sixth-order cumulants need to be truncated. The accuracy is also limited by the levels of nonlinearity of the oscillators and impulse arrival rate.

The exponential-polynomial closure (EPC) method was proposed for the PDF solution of nonlinear oscillators under Gaussian white noises [27–31]. In this paper, we extend the EPC method to obtain the approximate PDF solution of the generalized FPK equation for the case of Poisson impulses. Duffing oscillators, van der Pol oscillators, and an additional nonlinear oscillator excited by external and parametric Poisson impulses are studied to evaluate the effectiveness of the proposed approach. Oscillators with different nonlinear stiffness, nonlinear damping, and different types of parametric excitation, for example, parametric excitation on displacement or on velocity, are investigated. The numerical results show good agreement with Monte Carlo simulations for all the cases, even in the tail regions of the PDF solutions, which is of significance for reliability analyses.

II. Problem Formulation

Consider the following nonlinear stochastic oscillator:

$$\ddot{X} + h_0(X, \dot{X}) = h_j(X, \dot{X}) W_j(t) \qquad j = 1, 2, \ldots, N \tag{1}$$

where $X \in R$ and $\dot{X} \in R$ are stochastic processes; R denotes real space; h_0 and h_j are functions of X and $\dot{X}$, and their functional forms are assumed to be deterministic; and W_j is the jth Poisson white noise. The white noises are assumed to be independent of each other. A repeated subscript in a product indicates a summation over all the terms. The white noise is defined as

$$W_j(t) = \sum_{k=1}^{N(T)} Y_{jk} \delta(t - \tau_k) \tag{2}$$

where $N(T)$ is the total number of pulses that arrive in the time interval $(-\infty, T]$. It is assumed to yield Poisson law with a constant impulse arrival rate λ_j. Y_{jk} is the impulse amplitude of the kth pulse arriving at the time τ_k for $W_j(t)$. These amplitudes are independent and identically distributed (i.i.d.) random variables with zero mean and also independent of the pulse arrival time τ_k. $\delta(t)$ is the Dirac delta function. Setting $X = x_1$ and $\dot{X} = x_2$, Eq. (1) can be expressed as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h_0(x_1, x_2) + h_j(x_1, x_2) W_j(t) \end{cases} \tag{3}$$

The response vector $\{x_1, x_2\}^T$ is Markovian and can be stationary if the impulse arrival rate is constant with i.i.d. impulse amplitudes. The stationary PDF of the response is governed by the following generalized FPK equation [7]:

$$\begin{aligned} &- x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2} \left\{ \left(h_0 - \frac{1}{2} \lambda_j E[Y_j^2] h_j \frac{\partial h_j}{\partial x_2} \right) p \right\} \\ &+ \frac{1}{2!} \lambda_j E\left[Y_j^2\right] \frac{\partial^2}{\partial x_2^2} \left(h_j^2 p \right) - \frac{1}{3!} \lambda_j E\left[Y_j^3\right] \frac{\partial^3}{\partial x_2^3} (h_j^3 p) \\ &+ \frac{1}{4!} \lambda_j E\left[Y_j^4\right] \frac{\partial^4}{\partial x_2^4} \left(h_j^4 p \right) + \cdots = 0 \end{aligned} \tag{4}$$

where $E[\bullet]$ denotes the expectation of $(\bullet)$. Normally, only the terms up to the fourth-order derivative are retained for analysis because the contribution of high-order terms is small to the whole equation. It is assumed that the PDF $p(x_1, x_2)$ of the stationary response of the oscillator is subjected to the following conditions:

$$\begin{cases} p(x_1, x_2) \geq 0 & (x_1, x_2) \in R^2 \\ \lim\limits_{x_i \to \pm\infty} p(x_1, x_2) = 0 & i = 1, 2 \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 = 1 \end{cases} \tag{5}$$

To fulfill the conditions (5), an approximate PDF $\tilde{p}(x_1, x_2; \mathbf{a})$ solution to Eq. (4) is assumed to be [27–31]

$$\tilde{p}(x_1, x_2; \mathbf{a}) = c e^{Q_n(x_1, x_2; \mathbf{a})} \tag{6}$$

where c is a normalization constant and $\mathbf{a}$ is an unknown parameter vector containing N_p parameters. The polynomial $Q_n(x_1, x_2; \mathbf{a})$ is expressed as

$$Q_n(x_1, x_2; \mathbf{a}) = \sum_{i=1}^{n} \sum_{j=0}^{i} a_{ij} x_1^{i-j} x_2^j \tag{7}$$

which is an nth degree polynomial in x_1 and x_2. To fulfill the second requirement of the conditions (5), it is required that

$$Q_n(x_1, x_2; \mathbf{a}) = -\infty, \qquad (x_1, x_2) \notin \Omega \tag{8}$$

where $\Omega = [m_1 - c_1\sigma_1, m_1 + d_1\sigma_1] \times [m_2 - c_2\sigma_2, m_2 + d_2\sigma_2] \subset R^2$ in which m_i and σ_i denote the mean value and standard deviation of x_i $(i = 1, 2)$, respectively. The values of m_i and σ_i can be determined approximately by the EQL procedure or the Gaussian closure method for the case of Gaussian white noise. c_i and d_i are positive constants. We can select the values of c_i and d_i to locate $m_i - c_i\sigma_i$ and $m_i + d_i\sigma_i$ in the tails of the PDF of x_i. For instance, we normally set the values of c_i and d_i to be either 3 or 4. Equation (8) means that the approximate PDF is assumed to be valid within Ω.

Because $\tilde{p}(x_1, x_2; \mathbf{a})$ is only an approximation of $p(x_1, x_2)$, it leads to the following residual error by replacing $p(x_1, x_2)$ by $\tilde{p}(x_1, x_2; \mathbf{a})$ in Eq. (4) with the terms above the fourth-order derivative neglected:

$$\begin{aligned} \Delta(x_1, x_2; \mathbf{a}) = &- x_2 \frac{\partial \tilde{p}}{\partial x_1} + \frac{\partial}{\partial x_2} \left\{ \left(h_0 - \frac{1}{2} \lambda_j E\left[Y_j^2\right] h_j \frac{\partial h_j}{\partial x_2} \right) \tilde{p} \right\} \\ &+ \frac{1}{2!} \lambda_j E\left[Y_j^2\right] \frac{\partial^2}{\partial x_2^2} \left(h_j^2 \tilde{p} \right) - \frac{1}{3!} \lambda_j E\left[Y_j^3\right] \frac{\partial^3}{\partial x_2^3} \left(h_j^3 \tilde{p} \right) \\ &+ \frac{1}{4!} \lambda_j E\left[Y_j^4\right] \frac{\partial^4}{\partial x_2^4} \left(h_j^4 \tilde{p} \right) \end{aligned} \tag{9}$$

Substituting Eq. (6) into Eq. (9) and considering the following relations

$$\frac{\partial \tilde{p}}{\partial x_1} = \frac{\partial Q_n}{\partial x_1} \tilde{p} \tag{10a}$$

$$\frac{\partial \tilde{p}}{\partial x_2} = \frac{\partial Q_n}{\partial x_2} \tilde{p} \tag{10b}$$

$$\frac{\partial^2 \tilde{p}}{\partial x_2^2} = \left[\left(\frac{\partial Q_n}{\partial x_2} \right)^2 + \frac{\partial^2 Q_n}{\partial x_2^2} \right] \tilde{p} \tag{10c}$$

$$\frac{\partial^3 \tilde{p}}{\partial x_2^3} = \left[\left(\frac{\partial Q_n}{\partial x_2} \right)^3 + 3 \frac{\partial Q_n}{\partial x_2} \frac{\partial^2 Q_n}{\partial x_2^2} + \frac{\partial^3 Q_n}{\partial x_2^3} \right] \tilde{p} \tag{10d}$$

$$\begin{aligned} \frac{\partial^4 \tilde{p}}{\partial x_2^4} = &\left[\left(\frac{\partial Q_n}{\partial x_2} \right)^4 + 6 \left(\frac{\partial Q_n}{\partial x_2} \right)^2 \frac{\partial^2 Q_n}{\partial x_2^2} + 3 \left(\frac{\partial^2 Q_n}{\partial x_2^2} \right)^2 \right. \\ &\left. + 4 \frac{\partial Q_n}{\partial x_2} \frac{\partial^3 Q_n}{\partial x_2^3} + \frac{\partial^4 Q_n}{\partial x_2^4} \right] \tilde{p} \end{aligned} \tag{10e}$$

we obtain the following residual error

$$\Delta(x_1, x_2; \mathbf{a}) = F(x_1, x_2; \mathbf{a})\tilde{p}(x_1, x_2; \mathbf{a}) \tag{11}$$

where $F(x_1, x_2; \mathbf{a})$ is a function of $Q_n(x_1, x_2; \mathbf{a})$. Because $\tilde{p}(x_1, x_2; \mathbf{a})$ cannot be zero and $F(x_1, x_2; \mathbf{a})$ is not zero in general, another set of mutually independent functions $H_s(x_1, x_2)$ that span space R^{N_p} can be introduced to make the projection of $F(x_1, x_2; \mathbf{a})$ on R^{N_p} vanish. It leads to

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x_1, x_2; \mathbf{a}) H_s(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0 \qquad s = 1, 2, \ldots, N_p \tag{12}$$

Select $H_s(x_1, x_2)$ as

$$H_s(x_1, x_2) = x_1^{k-l} x_2^{l} f_1(x_1) f_2(x_2) \tag{13}$$

where $k = 1, 2, \ldots, n$; $l = 0, 1, 2, \ldots, k$; $s = (k+2)(k-1)/2 + l + 1$; and $N_p = N(N+3)/2$. The right-hand side of Eq. (13) is a single term and the repeated superscript l does not mean a summation of more terms. As a result, N_p nonlinear algebraic equations in terms of N_p unknowns are formulated. The algebraic equations can be solved with any available method to determine the unknowns. Numerical experience shows that a convenient and effective choice for $f_1(x_1)$ and $f_2(x_2)$ is the PDF of x_1 and x_2 obtained with the EQL method or the Gaussian closure method under Gaussian white noise with the intensity $\lambda_j E[Y_j^2]$ as follows:

$$f_1(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{x_1^2}{2\sigma_1^2}\right\} \tag{14}$$

$$f_2(x_2) = \frac{1}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{x_2^2}{2\sigma_2^2}\right\} \tag{15}$$

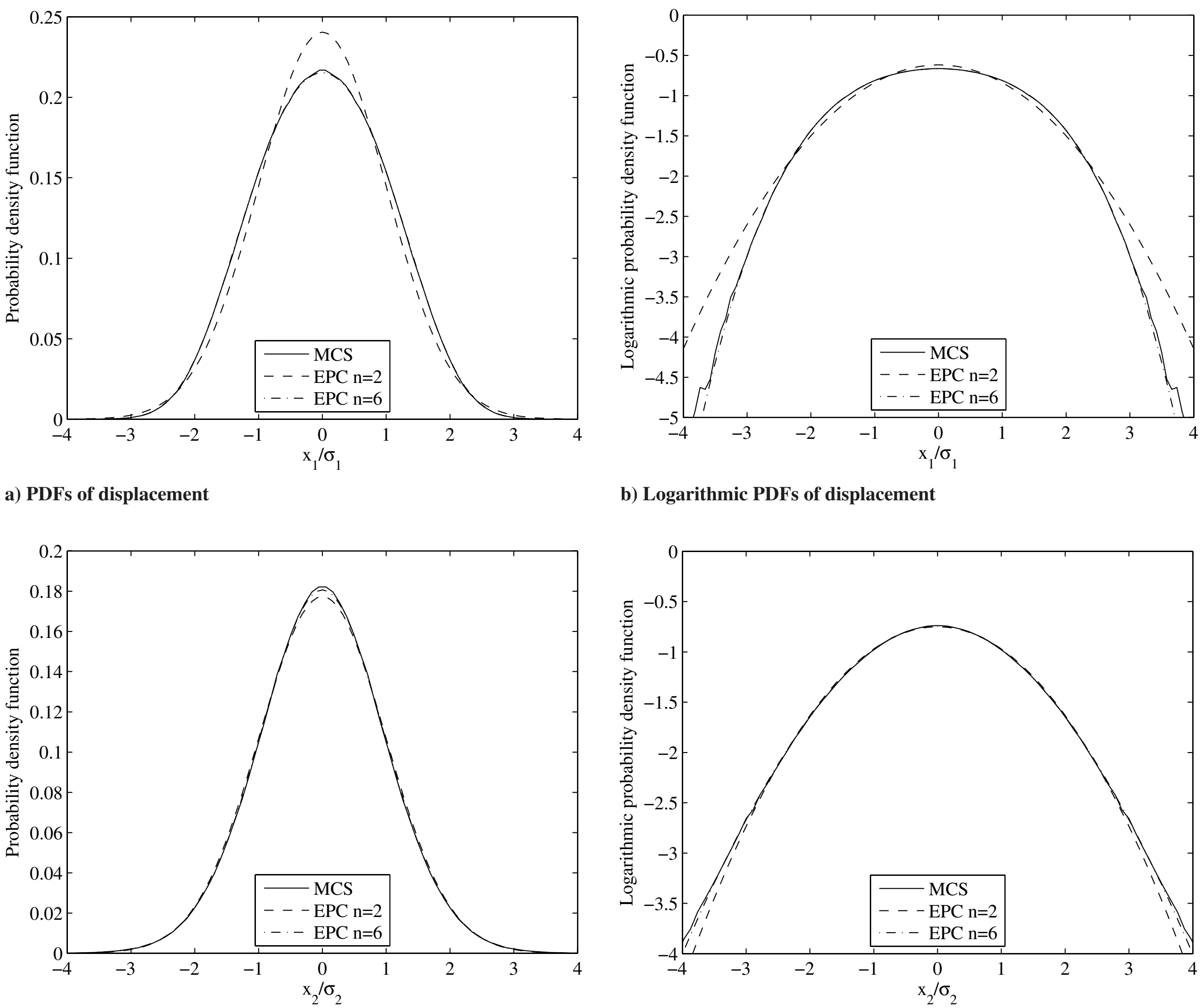
Because of the particular choice of $f_1(x_1)$ and $f_2(x_2)$, the integration in Eq. (12) can be easily evaluated by taking into account the relationships between the higher- and lower-order moments of Gaussian stochastic processes.

III. Illustrative Examples

To evaluate the effectiveness of the aforementioned solution procedure, we conduct a numerical analysis on some nonlinear oscillators excited by both external and parametric Poisson impulses. Results from the Monte Carlo simulation (MCS) are used to evaluate the effectiveness of the EPC method. The sample size N is determined by the following Chebyshev's inequality [32]

$$N \geq \frac{p_f(1-p_f)}{\delta\varepsilon^2} \tag{16}$$

where p_f is the probability to be estimated; δ and ε satisfy



a) PDFs of displacement

b) Logarithmic PDFs of displacement

c) PDFs of velocity

d) Logarithmic PDFs of velocity

Fig. 1 Response PDFs of the oscillator in Sec. III.A.

$$P(|\bar{p}_f - p_f| < \varepsilon) \geq 1 - \delta \tag{17}$$

where $\bar{p}_f$ is the estimated value of p_f, ε is the error tolerance, and $1-\delta$ is the confidence level. If the probability value to be estimated is 10^{-4} with the relative tolerance being 10% ($\varepsilon = 10^{-5}$) and the confidence level being 95% ($\delta = 0.05$), then the required sample size is 2×10^7 obtained from Eq. (16). This sample size is used for the Monte Carlo simulation in the following numerical analysis.

As a convention, "EPC ($n = \bullet$)" means "the EPC method with the polynomial degree being $\bullet$" in the following discussion.

A. Duffing Oscillator with Slight Nonlinearity and Parametric Excitation on Displacement

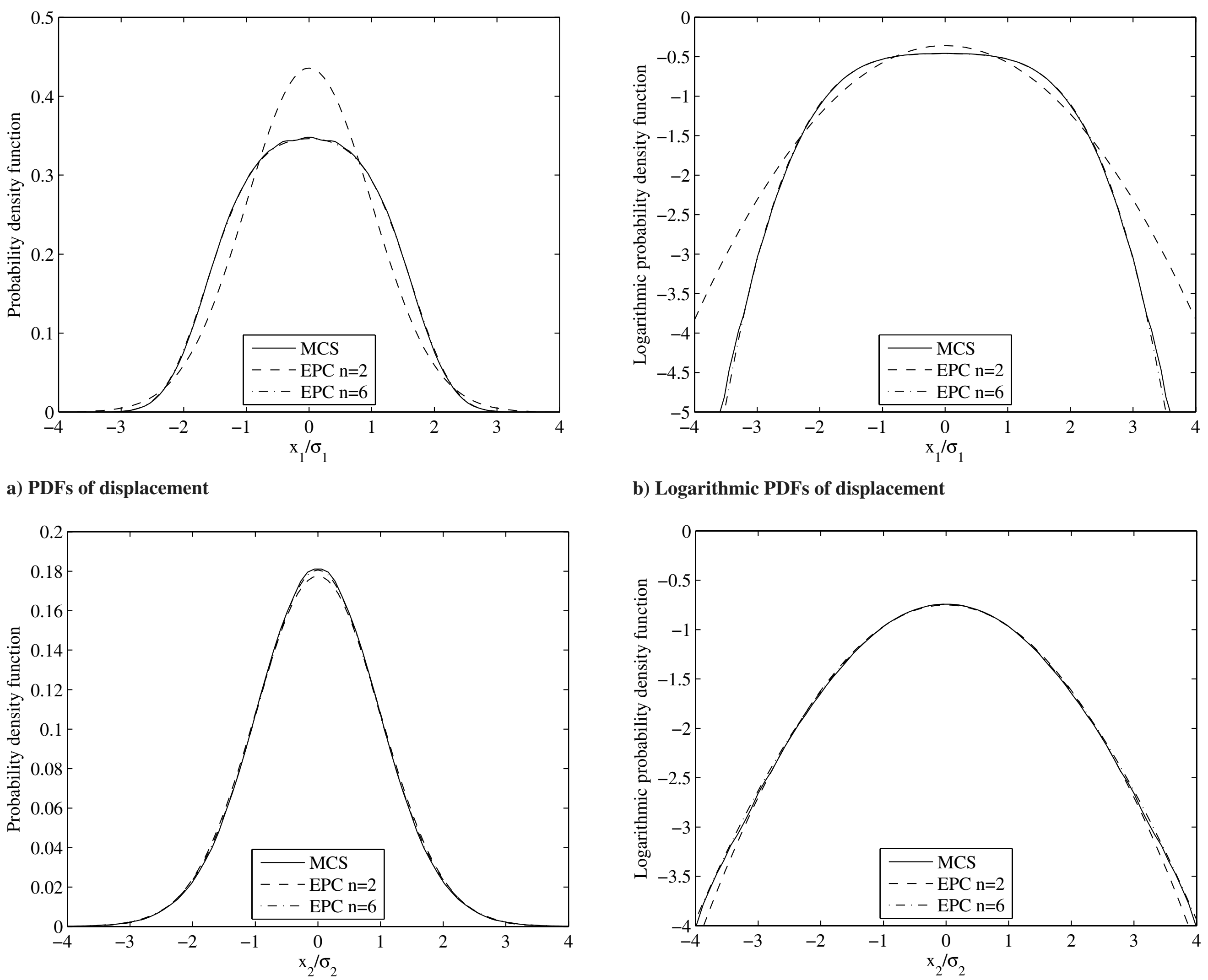
The first example is about a Duffing oscillator excited by both external and parametric Poisson white noise excitations. The multiplicative term of excitation on the displacement is considered and the oscillator is expressed by Eq. (18):

$$\ddot{X} + 2\zeta\omega_0\dot{X} + \omega_0^2(X + \varepsilon_1 X^3) = W_1(t) + \beta_1 X W_2(t) \tag{18}$$

in which the parameters are set with the following values: $\zeta = 0.05$, $\omega_0 = 1$, $\varepsilon_1 = 0.1$, $\beta_1 = 0.1$, $\lambda_1 E[Y_1^2] = \lambda_2 E[Y_2^2] = 1.0$, and $\lambda_1 = \lambda_2 = 2.0$. Y_1 and Y_2 are Gaussian with zero mean. $W_1(t)$ and $W_2(t)$ are independent of each other. In Eq. (1), $h_0 = 2\zeta\omega_0\dot{X} + \omega_0^2(X + \varepsilon_1 X^3)$, $h_1 = 1.0$, and $h_2 = \beta_1 X$. The value of ε_1 being 0.1 means that the nonlinearity of stiffness is slight. The nonlinear algebraic equations in terms of N_p unknowns in **a** can be formulated with Eq. (12). In this case, the expression of $F(x_1, x_2; \mathbf{a})$ in Eq. (12) is

$$\begin{aligned} F(x_1, x_2; \mathbf{a}) &= -x_2\frac{\partial Q_n}{\partial x_1} + \frac{\partial h_0}{\partial x_2} + h_0\frac{\partial Q_n}{\partial x_2} \\ &+ \frac{1}{2!}\left(\lambda_1 E\left[Y_1^2\right] + \lambda_2 E\left[Y_2^2\right]\beta_1^2 x_1^2\right)\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^2 + \frac{\partial^2 Q_n}{\partial x_2^2}\right] \\ &+ \frac{1}{4!}\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^4 + 6\left(\frac{\partial Q_n}{\partial x_2}\right)^2\frac{\partial^2 Q_n}{\partial x_2^2} + 3\left(\frac{\partial^2 Q_n}{\partial x_2^2}\right)^2 \right. \\ &\left. + 4\frac{\partial Q_n}{\partial x_2}\frac{\partial^3 Q_n}{\partial x_2^3} + \frac{\partial^4 Q_n}{\partial x_2^4}\right]\left(\lambda_1 E\left[Y_1^4\right] + \lambda_2 E\left[Y_2^4\right]\beta_1^4 x_1^4\right) \end{aligned} \tag{19}$$

Figures 1a and 1b present the PDF and logarithmic PDF solutions of displacement and velocity obtained with EPC ($n = 2, 6$) and MCS, respectively. Numerical analysis shows that the results obtained with EPC ($n = 2$) are very close to those obtained with the EQL method or the Gaussian closure method under Gaussian white noises with the same intensities as Poisson white noises. The curves obtained with these methods overlap each other in the plots of PDF. Therefore, the results obtained with EPC ($n = 2$) shown in the figures represent the results obtained with the EQL method. This



a) PDFs of displacement

b) Logarithmic PDFs of displacement

c) PDFs of velocity

d) Logarithmic PDFs of velocity

Fig. 2 Response PDFs of the oscillator in Sec. III.B.

observation is also applicable to the other cases in the following examples. Meanwhile, it is also observed that the PDF of displacement obtained with EPC ($n = 2$) deviates significantly from the simulated one. As the polynomial degree n increases to 6, the PDF of displacement obtained with EPC ($n = 6$) agrees well with the simulated one. The tail behavior of the PDFs of displacement obtained with various methods is shown in Fig. 1b. It is seen that the PDF of displacement obtained with EPC ($n = 6$) is also much improved compared with that obtained with EPC ($n = 2$). The PDFs and logarithmic PDFs of velocity obtained with various methods are shown in Figs. 1c and 1d. We observe from these figures that the PDFs obtained with EPC ($n = 2$) and EPC ($n = 6$) are close to each other, but EPC ($n = 6$) still provides an improved result. It means that the PDF of velocity is close to the Gaussian PDF in this case.

B. Duffing Oscillator with High Nonlinearity and Parametric Excitation on Displacement

In this case, the oscillator, the expression of $F(x_1, x_2; \mathbf{a})$ in Eq. (12), and the values of the parameters are same as those in Sec. III.A except that $\varepsilon_1 = 2.0$ is used for high nonlinearity.

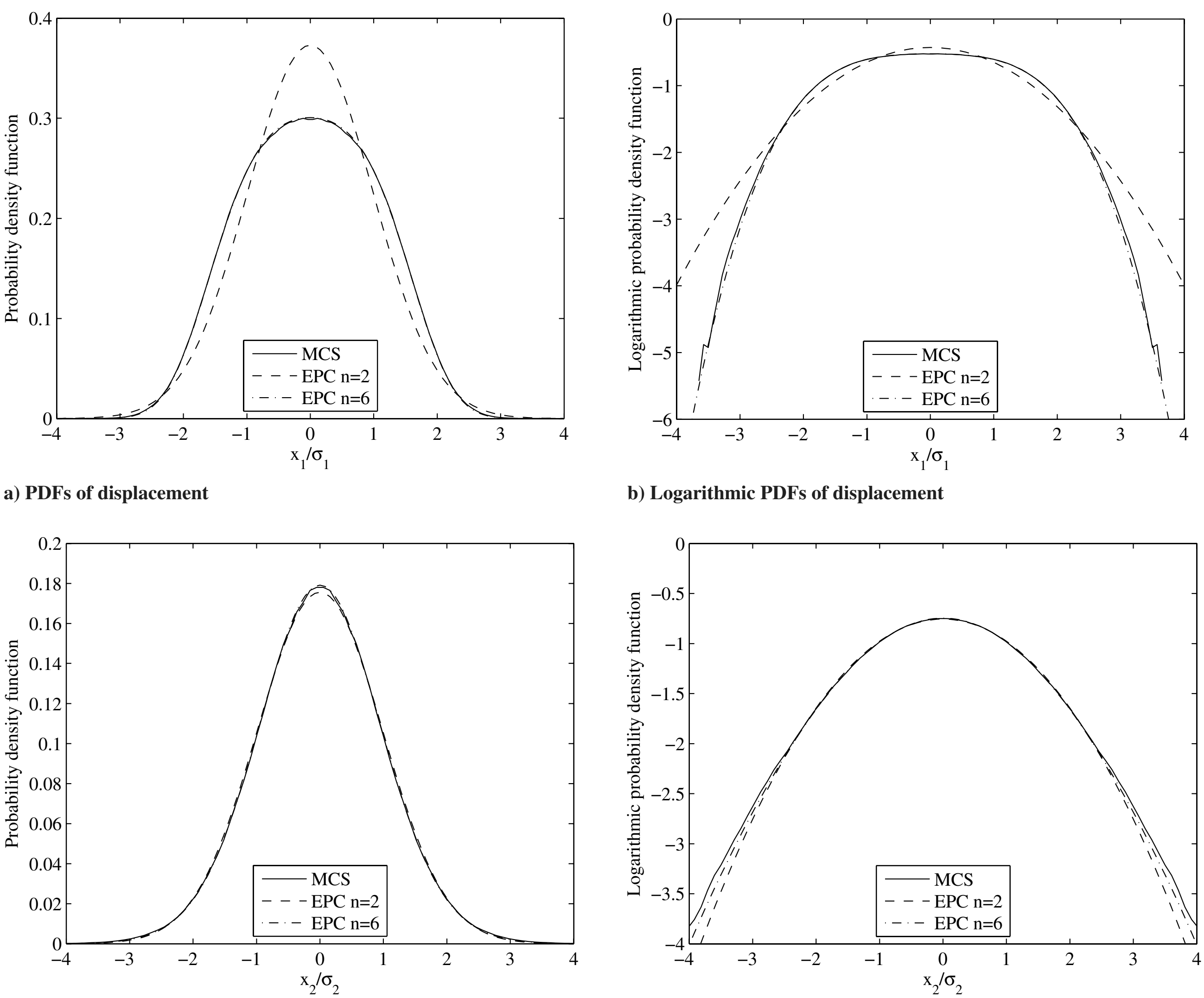
From Figs. 2a and 2b, we observe that the PDF of displacement obtained with EPC ($n = 2$) deviates a lot from the simulated one, but the PDF of displacement obtained with EPC ($n = 6$) is much closer to the simulated result. Even in the tails of the PDF of displacement, the result obtained with EPC ($n = 6$) is still close to the simulated one. The PDFs of velocity obtained with various methods are shown and compared in Figs. 2c and 2d, with the same conclusion being drawn as those in Sec. III.A. Hence, we see that the presented solution procedure is also valid for the oscillators with a high stiffness nonlinearity.

C. Duffing Oscillator with Parametric Excitations on Displacement and Velocity

This example is about a Duffing oscillator excited by external Poisson white noise excitation and parametric Poisson white noise excitations acting on displacement as well as on velocity. The nonlinear stochastic oscillator is given by

$$\ddot{X} + 2\zeta\omega_0\dot{X} + \omega_0^2(X + \varepsilon_1 X^3) = W_1(t) + \beta_1 X W_2(t) + \beta_2 \dot{X} W_3(t) \tag{20}$$

in which the parameters are set with the following values: $\zeta = 0.05$, $\omega_0 = 1$, $\varepsilon_1 = 1.0$, $\beta_1 = 0.2$, $\beta_2 = 0.1$, $\lambda_1 E[Y_1^2] = \lambda_2 E[Y_2^2] = 1.0$, $\lambda_3 E[Y_3^2] = 0.1$, and $\lambda_1 = \lambda_2 = \lambda_3 = 2.0$. Y_1, Y_2, and Y_3 are Gaussian with zero mean. $W_1(t)$, $W_2(t)$, and $W_3(t)$ are independent of one another. In Eq. (1), $h_0 = 2\zeta\omega_0\dot{X} + \omega_0^2(X + \varepsilon_1 X^3)$, $h_1 = 1.0$,



a) PDFs of displacement

b) Logarithmic PDFs of displacement

c) PDFs of velocity

d) Logarithmic PDFs of velocity

Fig. 3 Response PDFs of the oscillator in Sec. III.C.

$h_2 = \beta_1 X$, and $h_3 = \beta_2 \dot{X}$. In this case, the expression of $F(x_1, x_2; \mathbf{a})$ in Eq. (12) is as follows:

$$
\begin{aligned}
F(x_1, x_2; \mathbf{a}) = &-x_2 \frac{\partial Q_n}{\partial x_1} + \frac{\partial h_0}{\partial x_2} + \frac{1}{2}\lambda_3 E\left[Y_3^2\right]\beta_2^2 + \frac{24}{4!}\lambda_3 E\left[Y_3^4\right]\beta_2^4 \\
&+ \left(h_0 + \frac{3}{2!}\lambda_3 E\left[Y_3^2\right]\beta_2^2 x_2 + \frac{96}{4!}\lambda_3 E\left[Y_3^4\right]\beta_2^4 x_2\right)\frac{\partial Q_n}{\partial x_2} \\
&+ \left\{\frac{1}{2!}\lambda_1 E\left[Y_1^2\right] + \frac{1}{2!}\lambda_2 E\left[Y_2^2\right]\beta_1^2 x_1^2 + \frac{1}{2!}\lambda_3 E\left[Y_3^2\right]\beta_2^2 x_2^2 \right. \\
&\left. + \frac{72}{4!}\lambda_3 E\left[Y_3^4\right]\beta_2^4 x_2^2\right\}\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^2 + \frac{\partial^2 Q_n}{\partial x_2^2}\right] \\
&+ \frac{16}{4!}\lambda_3 E\left[Y_3^4\right]\beta_2^4 x_2^3\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^3 + 3\frac{\partial Q_n}{\partial x_2}\frac{\partial^2 Q_n}{\partial x_2^2} + \frac{\partial^3 Q_n}{\partial x_2^3}\right] \\
&+ \frac{1}{4!}\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^4 + 6\left(\frac{\partial Q_n}{\partial x_2}\right)^2 \frac{\partial^2 Q_n}{\partial x_2^2} \right. \\
&\left. + 3\left(\frac{\partial^2 Q_n}{\partial x_2^2}\right)^2 + 4\frac{\partial Q_n}{\partial x_2}\frac{\partial^3 Q_n}{\partial x_2^3} + \frac{\partial^4 Q_n}{\partial x_2^4}\right]\left(\lambda_1 E\left[Y_1^4\right] \right. \\
&\left. + \lambda_2 E\left[Y_2^4\right]\beta_1^4 x_1^4 + \lambda_3 E\left[Y_3^4\right]\beta_2^4 x_2^4\right)
\end{aligned}
\tag{21}
$$

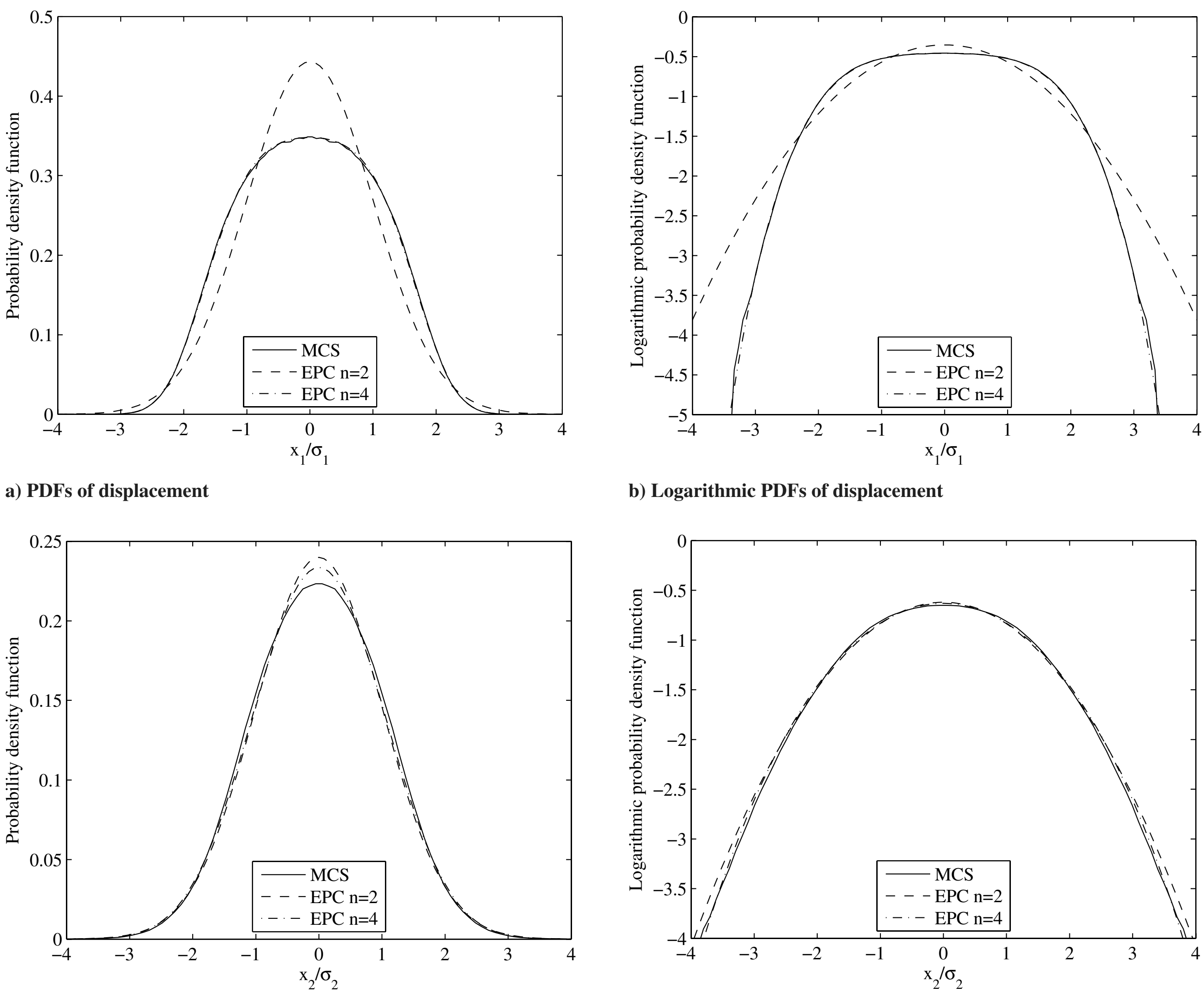
It is seen from Figs. 3a–3d that the PDFs obtained with EPC ($n = 6$) are still in good agreement with those obtained with MCS for both displacement and velocity, whereas the PDF of displacement obtained with EPC ($n = 2$) differs significantly from the simulated one. The difference is insignificant among the PDFs of velocity obtained with EPC ($n = 2, 6$) and MCS, but the result obtained with EPC ($n = 6$) is still improved compared with that obtained with EPC ($n = 2$).

D. Nonlinear Oscillator with Nonlinear Stiffness, Nonlinear Damping, and Parametric Excitation on Velocity

The nonlinear stochastic oscillator is expressed as follows:

$$
\ddot{X} + 2\zeta\omega_0(\dot{X} + \varepsilon_1 \dot{X}^3) + \omega_0^2(X + \varepsilon_2 X^3) = W_1(t) + \beta_1 \dot{X} W_2(t) \tag{22}
$$

in which the parameters are set with the following values: $\zeta = 0.05$, $\omega_0 = 1$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 1.0$, $\beta_1 = 0.1$, $\lambda_1 E[Y_1^2] = 1.0$, $\lambda_2 E[Y_2^2] = 0.1$, $\lambda_1 = 1.0$, and $\lambda_2 = 2.0$. Y_1 and Y_2 are Gaussian with zero mean. $W_1(t)$ and $W_2(t)$ are independent of each other. In Eq. (1), $h_0 = 2\zeta\omega_0(\dot{X} + \varepsilon_1 \dot{X}^3) + \omega_0^2(X + \varepsilon_2 X^3)$, $h_1 = 1.0$, and $h_2 = \beta_1 \dot{X}$. In this case, the expression of $F(x_1, x_2; \mathbf{a})$ in Eq. (12) is as follows:



a) PDFs of displacement

b) Logarithmic PDFs of displacement

c) PDFs of velocity

d) Logarithmic PDFs of velocity

Fig. 4 Response PDFs of the oscillator in Sec. III.D.

$$
\begin{aligned}
F(x_1, x_2; \mathbf{a}) = {} & -x_2 \frac{\partial Q_n}{\partial x_1} + \frac{\partial h_0}{\partial x_2} + \frac{1}{2}\lambda_2 E\Big[Y_2^2\Big]\beta_1^2 + \frac{24}{4!}\lambda_2 E\Big[Y_2^4\Big]\beta_1^4 \\
& + \left(h_0 + \frac{3}{2!}\lambda_2 E\Big[Y_2^2\Big]\beta_1^2 x_2 + \frac{96}{4!}\lambda_2 E\Big[Y_2^4\Big]\beta_1^4 x_2\right)\frac{\partial Q_n}{\partial x_2} \\
& + \left\{\frac{1}{2!}\lambda_1 E\Big[Y_1^2\Big] + \frac{1}{2!}\lambda_2 E\Big[Y_2^2\Big]\beta_1^2 x_2^2\right. \\
& \left. + \frac{72}{4!}\lambda_2 E\Big[Y_2^4\Big]\beta_1^4 x_2^2\right\}\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^2 + \frac{\partial^2 Q_n}{\partial x_2^2}\right] \\
& + \frac{16}{4!}\lambda_2 E\Big[Y_2^4\Big]\beta_1^4 x_2^3\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^3 + 3\frac{\partial Q_n}{\partial x_2}\frac{\partial^2 Q_n}{\partial x_2^2} + \frac{\partial^3 Q_n}{\partial x_2^3}\right] \\
& + \frac{1}{4!}\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^4 + 6\left(\frac{\partial Q_n}{\partial x_2}\right)^2\frac{\partial^2 Q_n}{\partial x_2^2} + 3\left(\frac{\partial^2 Q_n}{\partial x_2^2}\right)^2\right. \\
& \left. + 4\frac{\partial Q_n}{\partial x_2}\frac{\partial^3 Q_n}{\partial x_2^3} + \frac{\partial^4 Q_n}{\partial x_2^4}\right]\left(\lambda_1 E\Big[Y_1^4\Big] + \lambda_2 E\Big[Y_2^4\Big]\beta_1^4 x_2^4\right) \qquad (23)
\end{aligned}
$$

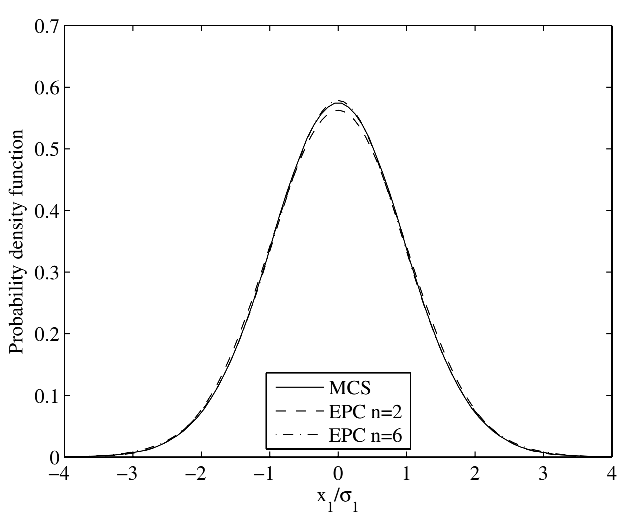
The results obtained with various methods are compared and shown in Figs. 4a–4d. In this case, the PDF of displacement obtained with EPC ($n = 2$) differs significantly from the simulated one, as shown in Fig. 4a. Meanwhile, the PDF of displacement obtained with EPC ($n = 4$) agrees well with the result obtained with MCS. This can also be observed in the tail regions of the PDF, as shown in Fig. 4b. As for the PDF of velocity, little difference can be observed among the results obtained with EPC ($n = 2, 4$) and MCS, as shown in Fig. 4c. In the tail regions, the result obtained with EPC ($n = 4$) is still improved compared with the result obtained with EPC ($n = 2$), as shown in Fig. 4d, though the PDF of velocity is close to the Gaussian PDF.

E. Van der Pol Oscillator with Slight Nonlinearity in Damping and Parametric Excitation on Velocity

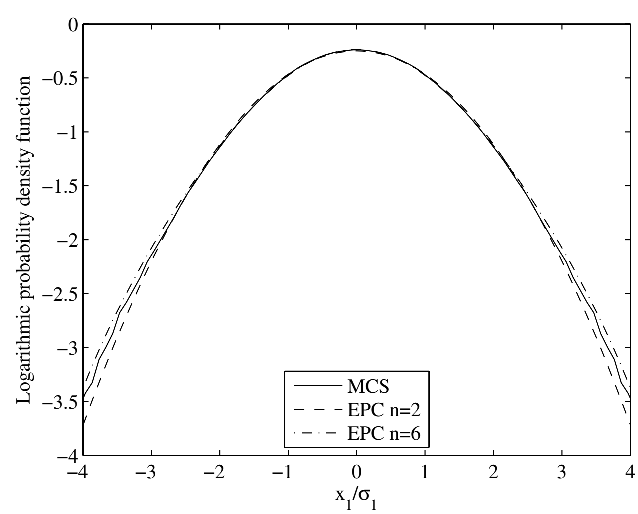
The following van der Pol oscillator with parametric excitation on velocity is investigated:

$$\ddot{X} + 2\zeta\omega_0\dot{X} + \varepsilon_1 X^2\dot{X} + \omega_0^2 X = W_1(t) + \beta_1\dot{X}W_2(t) \qquad (24)$$

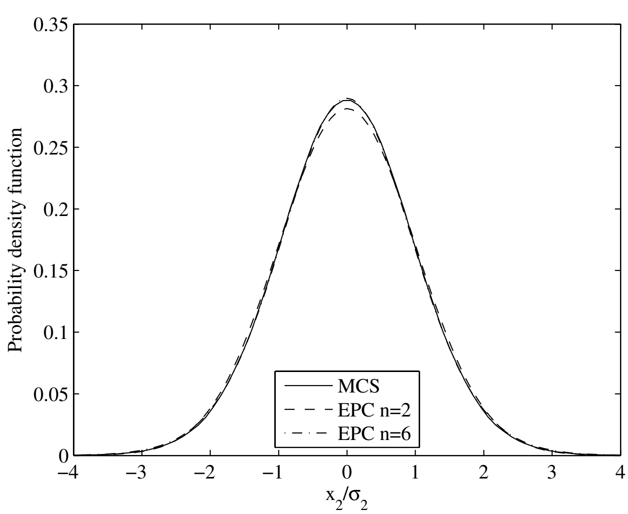
in which the parameters are set with the following values: $\zeta = 0.05$, $\omega_0 = 2$, $\varepsilon_1 = 0.1$, $\beta_1 = 0.1$, $\lambda_1 E[Y_1^2] = 1.0$, $\lambda_2 E[Y_2^2] = 0.1$, and $\lambda_1 = \lambda_2 = 1.0$. Y_1 and Y_2 are Gaussian with zero mean. $W_1(t)$ and $W_2(t)$ are independent of each other. In Eq. (1), $h_0 = 2\zeta\omega_0\dot{X} + \varepsilon_1 X^2\dot{X} + \omega_0^2 X$, $h_1 = 1.0$, and $h_2 = \beta_1\dot{X}$. The slight nonlinearity in damping with $\varepsilon_1 = 0.1$ is considered to examine the effectiveness of the solution procedure. In this case, the expression of $F(x_1, x_2; \mathbf{a})$ is same as Eq. (23).



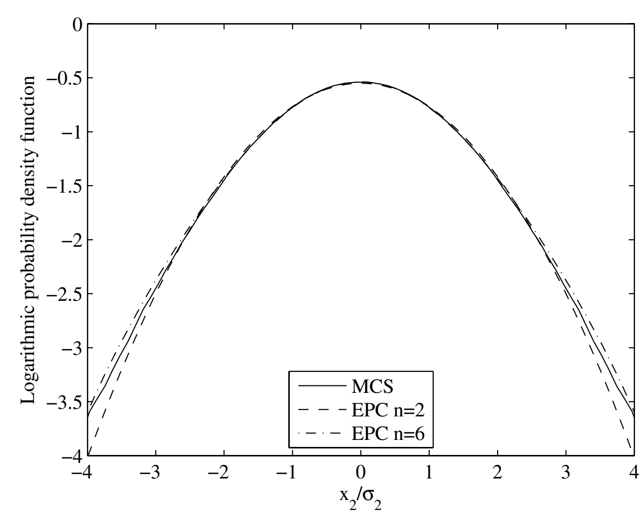
a) PDFs of displacement



b) Logarithmic PDFs of displacement

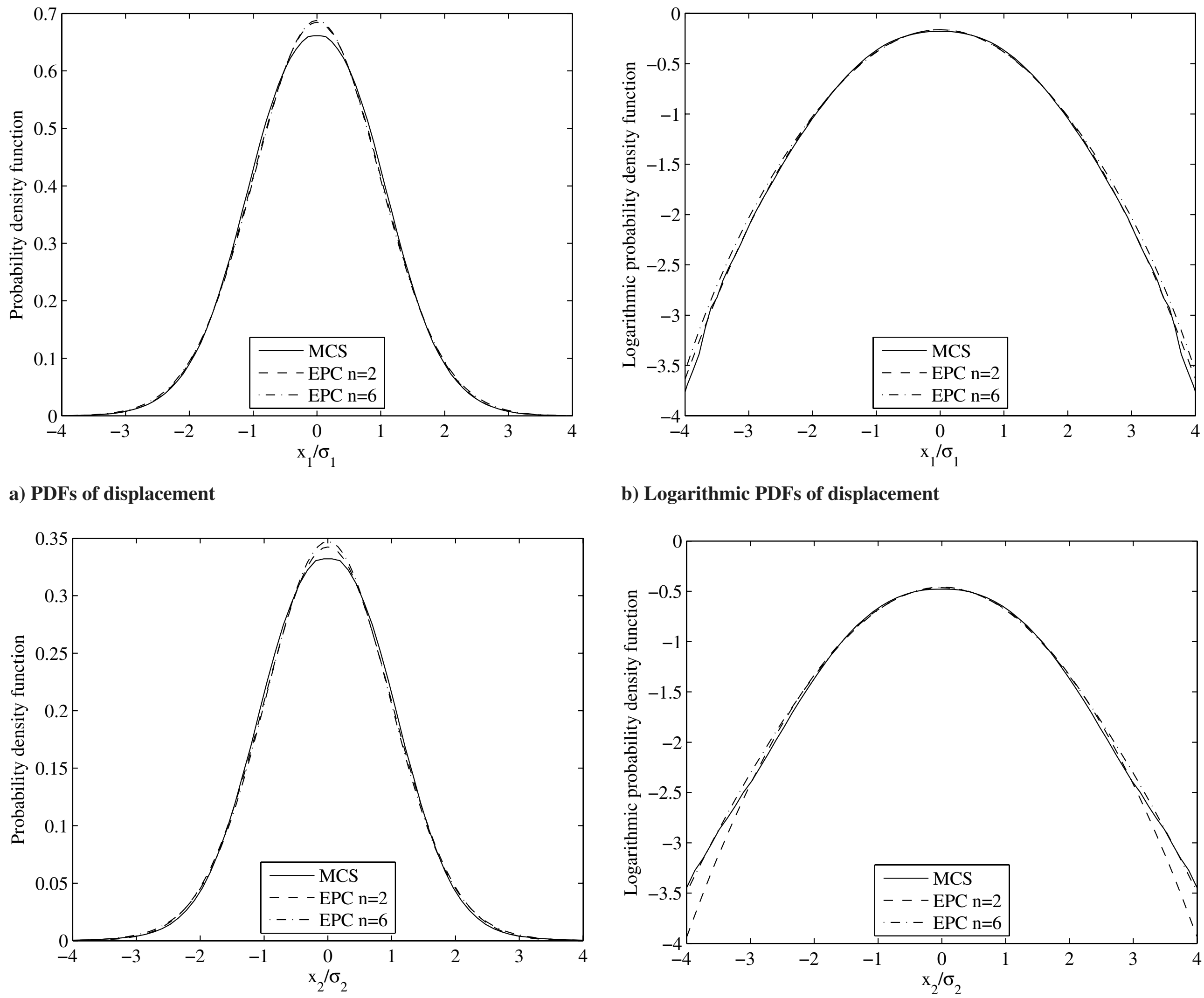


c) PDFs of velocity



d) Logarithmic PDFs of velocity

Fig. 5 Response PDFs of the oscillator in Sec. III.E.



a) PDFs of displacement

b) Logarithmic PDFs of displacement

c) PDFs of velocity

d) Logarithmic PDFs of velocity

Fig. 6 Response PDFs of the oscillator in Sec. III.F.

The PDFs of displacement and velocity obtained with various methods are shown in Figs. 5a–5d. From Figs. 5a and 5b, we observe that the PDF of displacement is close to the Gaussian PDF though it is not Gaussian, whereas the EPC ($n = 6$) provides an improved result compared with the result from EPC ($n = 2$). We also observe that the PDF of velocity deviates significantly from being Gaussian in this case, but the PDF of velocity obtained with EPC ($n = 6$) still agrees well with that from MCS even in the tails of the PDF, as shown in Fig. 5d.

F. Van der Pol Oscillator with High Nonlinearity in Damping and Parametric Excitation on Velocity

In this example, the van der Pol oscillator in Sec. III.E is used except that $\varepsilon_1 = 0.5$ so that a high nonlinearity in damping can be considered. Comparing Figs. 5a–5d with Figs. 6a–6d, we observe that the PDF of displacement in the case of a high nonlinearity in damping is more Gaussian than that in the case of a slight nonlinearity in damping, whereas the PDF of velocity in the case of a high nonlinearity in damping deviates more from being Gaussian than that in the case of a slight nonlinearity in damping. In this case, EPC ($n = 6$) still provides improved results for the PDFs of displacement and velocity compared with EPC ($n = 2$).

IV. Conclusions

From this discussion, it can be concluded that the EPC method is applicable for the PDF analysis of nonlinear oscillators subjected to both external and parametric Poisson white noise excitations. With the polynomial degree n being either 4 or 6, the obtained PDFs of the response are very close to those obtained with MCS even in the tails of the PDFs, which is of significance in reliability analyses. The EPC method is effective for the oscillators with either a slight or high nonlinearity in stiffness and damping. Meanwhile, it is also observed that the tails of the PDFs obtained with the EQL method or the Gaussian closure method deviate significantly from the simulated results even for slightly nonlinear oscillators. Hence, both the EQL method and the Gaussian closure method are inadequate for the PDF solutions of nonlinear oscillators under parametric Poisson impulse excitations, especially with regard to the tails of the PDF of the response. In this case, the EPC method with the polynomial degree being either 4 or 6 can provide a potential tool for the PDF solutions of nonlinear oscillators.

Acknowledgments

The authors gratefully acknowledge the Research Committee of the University of Macau for providing financial support (grant nos. RG062/05-06S/KKP/FST, RG062/05-06S/07R/EGK/FST, and RG062/05-06S/08R/EGK/FST).

References

[1] Abou-Rayan, A. M., and Nayfeh, A. H., "Stochastic Responses of a Buckled Beam to External and Parametric Random Excitations," *Proceedings of the 34th AIAA/ASME/ASCE/AHS/ASC Structures,*

Structural Dynamics and Materials Conference, AIAA, Washington, D. C., 1993, pp. 1030–1040.

[2] Lin, Y. K., and Yang, J. N., "A Stochastic Theory of Fatigue Crack Propagation," *AIAA Journal*, Vol. 23, No. 1, 1985, pp. 117–124. doi:10.2514/3.8879; also AIAA Paper 83-0978, May 1983.

[3] Enneking, T. J., and Spencer, B. F., Jr., "Distribution of Crack Sizes—Solution of the Forward Kolmogorov Equation," *Proceedings of the 33rd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*, AIAA, Washington, D. C., 1992, pp. 2871–2878.

[4] Pandya, R. V. R., and Mashayek, F., "Probability Density Function Modeling of Evaporating Droplets Dispersed in Isotropic Turbulence," *AIAA Journal*, Vol. 39, No. 10, 2001, pp. 1909–1915. doi:10.2514/2.1180; also AIAA Paper 2001-0333, Jan. 2001.

[5] Grigoriu, M., "White Noise Processes," *Journal of Engineering Mechanics*, Vol. 113, No. 5, May 1987, pp. 757–765.

[6] Roberts, J. B., "System Response to Random Impulses," *Journal of Sound and Vibration*, Vol. 24, No. 1, 1972, pp. 23–34. doi:10.1016/0022-460X(72)90119-8

[7] Cai, G. Q., and Lin, Y. K., "Response Distribution of Non-Linear Systems Excited by Non-Gaussian Impulsive Noise," *International Journal of Non-Linear Mechanics*, Vol. 27, No. 6, 1992, pp. 955–967. doi:10.1016/0020-7462(92)90048-C

[8] Di Paola, M., and Pirrotta, A., "Direct Derivation of Corrective Terms in SDE Through Nonlinear Transformation on Fokker–Planck Equation," *Nonlinear Dynamics*, Vol. 36, Nos. 2–4, 2004, pp. 349–360. doi:10.1023/B:NODY.0000045511.89550.57

[9] Pirrotta, A., "Multiplicative Cases from Additive Cases: Extension of Kolmogorov-Feller Equation to Parametric Poisson White Noise Processes," *Probabilistic Engineering Mechanics*, Vol. 22, No. 2, 2007, pp. 127–135. doi:10.1016/j.probengmech.2006.08.005

[10] Vasta, M., "Exact Stationary Solution for a Class of Non-Linear Systems Driven by a Non-Normal Delta-Correlated Process," *International Journal of Non-Linear Mechanics*, Vol. 30, No. 4, 1995, pp. 407–418. doi:10.1016/0020-7462(95)00009-D

[11] Proppe, C., "The Wong-Zakai Theorem for Dynamical Systems with Parametric Poisson White Noise Excitation," *International Journal of Engineering Science*, Vol. 40, No. 10, 2002, pp. 1165–1178. doi:10.1016/S0020-7225(01)00087-8

[12] Proppe, C., "Exact Stationary Probability Density Functions for Non-Linear Systems Under Poisson White Noise Excitation," *International Journal of Non-Linear Mechanics*, Vol. 38, No. 4, 2003, pp. 557–564. doi:10.1016/S0020-7462(01)00084-1

[13] Köylüoğlu, H. U., Nielsen, S. R. K., and Iwankiewicz, R., "Reliability of Non-Linear Oscillators Subject to Poisson Driven Impulses," *Journal of Sound and Vibration*, Vol. 176, No. 1, 1994, pp. 19–33. doi:10.1006/jsvi.1994.1356

[14] Köylüoğlu, H. U., Nielsen, S. R. K., and Iwankiewicz, R., "Response and Reliability of Poisson-Driven Systems by Path Integration," *Journal of Engineering Mechanics*, Vol. 121, No. 1, 1995, pp. 117–130. doi:10.1061/(ASCE)0733-9399(1995)121:1(117)

[15] Köylüoğlu, H. U., Nielsen, S. R. K., and Çakmak, A. Ş., "Fast Cell-to-Cell Mapping (Path Integration) for Nonlinear White Noise and Poisson Driven Systems," *Structural Safety*, Vol. 17, No. 3, 1995, pp. 151–165. doi:10.1016/0167-4730(95)00006-P

[16] Iwankiewicz, R., and Nielsen, S. R. K., "Solution Techniques for Pulse Problems in Non-Linear Stochastic Dynamics," *Probabilistic Engineering Mechanics*, Vol. 15, No. 1, 2000, pp. 25–36. doi:10.1016/S0266-8920(99)00006-5

[17] Wojtkiewicz, S. F., Johnson, E. A., Bergman, L. A., Spencer, B. F., Jr., and Grigoriu, M., "Stochastic Response to Additive Gaussian and Poisson White Noises," *Proceedings of the 4th International Conference on Stochastic Structural Dynamics*, Balkema, Rotterdam, The Netherlands, 1999, pp. 53–60.

[18] Wojtkiewicz, S. F., Johnson, E. A., Bergman, L. A., Grigoriu, M., and Spencer, B. F., Jr., "Response of Stochastic Dynamical Systems Driven by Additive Gaussian and Poisson White Noise: Solution of a Forward Generalized Kolmogorov Equation by a Spectral Finite Difference Method," *Computer Methods in Applied Mechanics and Engineering*, Vol. 168, Nos. 1–4, 1999, pp. 73–89. doi:10.1016/S0045-7825(98)00098-X

[19] Tylikowski, A., and Marowski, W., "Vibration of a Non-Linear Single Degree of Freedom System Due to Poissonian Impulse Excitation," *International Journal of Non-Linear Mechanics*, Vol. 21, No. 3, 1986, pp. 229–238. doi:10.1016/0020-7462(86)90006-5

[20] Grigoriu, M., "Equivalent Linearization for Poisson White Noise Input," *Probabilistic Engineering Mechanics*, Vol. 10, No. 1, 1995, pp. 45–51. doi:10.1016/0266-8920(94)00007-8

[21] Sobiechowski, C., and Socha, L., "Statistical Linearization of the Duffing Oscillator Under Non-Gaussian External Excitation," *Journal of Sound and Vibration*, Vol. 231, No. 1, 2000, pp. 19–35. doi:10.1006/jsvi.1999.2668

[22] Proppe, C., "Equivalent Linearization of MDOF Systems Under External Poisson White Noise Excitation," *Probabilistic Engineering Mechanics*, Vol. 17, No. 4, 2002, pp. 393–399. doi:10.1016/S0266-8920(02)00036-X

[23] Proppe, C., "Stochastic Linearization of Dynamical Systems Under Parametric Poisson White Noise Excitation," *International Journal of Non-Linear Mechanics*, Vol. 38, No. 4, 2003, pp. 543–555. doi:10.1016/S0020-7462(01)00083-X

[24] Iwankiewicz, R., Nielsen, S. R. K., and Thoft-Christensen, P., "Dynamic Response of Non-Linear Systems to Poisson-Distributed Pulse Trains: Markov Approach," *Structural Safety*, Vol. 8, Nos. 1–4, 1990, pp. 223–238. doi:10.1016/0167-4730(90)90042-N

[25] Iwankiewicz, R., and Nielsen, S. R. K., "Dynamic Response of Non-Linear Systems to Poisson-Distributed Random Impulses," *Journal of Sound and Vibration*, Vol. 156, No. 3, 1992, pp. 407–423. doi:10.1016/0022-460X(92)90736-H

[26] Di Paola, M., and Falsone, G., "Non-Linear Oscillators Under Parametric and External Poisson Pulses," *Nonlinear Dynamics*, Vol. 5, No. 3, 1994, pp. 337–352. doi:10.1007/BF00045341

[27] Er, G. K., "A New Non-Gaussian Closure Method for the PDF Solution of Nonlinear Random Vibrations," *Proceedings of the 12th Engineering Mechanics Conference*, American Society of Civil Engineers, Reston, VA, 1998, pp. 1403–1406.

[28] Er, G. K., "An Improved Closure Method for Analysis of Nonlinear Stochastic Systems," *Nonlinear Dynamics*, Vol. 17, No. 3, 1998, pp. 285–297. doi:10.1023/A:1008346204836

[29] Er, G. K., and Iu, V. P., "Probabilistic Solutions to Nonlinear Random Ship Roll Motion," *Journal of Engineering Mechanics*, Vol. 125, No. 5, 1999, pp. 570–574. doi:10.1061/(ASCE)0733-9399(1999)125:5(570)

[30] Er, G. K., "The Probabilistic Solutions to Nonlinear Random Vibrations of Multi-Degree-of-Freedom Systems," *Journal of Applied Mechanics*, Vol. 67, No. 2, June 2000, pp. 355–359. doi:10.1115/1.1304842

[31] Er, G. K., and Iu, V. P., "Stochastic Response of Base-Excited Coulomb Oscillator," *Journal of Sound and Vibration*, Vol. 233, No. 1, 2000, pp. 81–92. doi:10.1006/jsvi.1999.2792

[32] Fishman, G. S., *Monte Carlo: Concepts, Algorithms, and Applications*, Springer–Verlag, New York, 1995, Chap. 2.

R. Kapania
Associate Editor